COMMUTATIVE MATRIX ALGEBRAS OF LENGTH $n-2$

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Definition. The *length of a finite system of generators* $\mathcal{S}$ for a finite-dimensional associative algebra $\mathcal{A}$ over an arbitrary field is defined as the least nonnegative integer $l(\mathcal{S})$ such that the words in $\mathcal{S}$ of length not exceeding $l(\mathcal{S})$ span this algebra (as a vector space). The maximum length for the systems of generators of an algebra is referred to as the *length of the algebra*, we denote it by $l(\mathcal{A})$ (see [1]).

The problem of computing the length of the full matrix algebra $M_n(\mathbb{F})$ over a field as a function of the matrix size n was stated by A. Paz in [2] and is still open in general. On the other hand, for a commutative subalgebra in the matrix algebra of order n the linear on n upper bound is known, namely over the field of complex numbers $\mathbb{C}$ it was established by A. Paz that its length is not greater than $n-1$. It was later proved in [3-4] that this bound also holds for commutative matrix subalgebras over arbitrary fields and that a commutative subalgebra $\mathcal{A}$ in the matrix algebra $M_n(\mathbb{F})$ satisfies the equation $l(\mathcal{A}) = n-1$ if and only if the algebra $\mathcal{A}$ is generated by a *non-derogatory* matrix C, i.e. by such a matrix $C \in M_n(\mathbb{F})$, that $\dim_{\mathbb{F}}(\langle C^0 = I_n,\ C,\ C^2, \ldots,\ C^{n-1}\rangle) = n$.

In the present talk we describe commutative subalgebras of the length $n-2$ in the algebra $M_n(\mathbb{F})$, i.e. of the length closest to maximal, over fields which contain at least $n+1$ elements.

The following theorem shows that this question can be reduced to the case of nilpotent commutative subalgebras:

Theorem 1. *Let $\mathbb{F}$ be an algebraically closed field and let $n \in \mathbb{N}$, $n \geq 2$. Consider a commutative subalgebra $\mathcal{A}$ in $M_n(\mathbb{F})$ of the length $l(\mathcal{A}) = n-2$. Then there exist a number $m \in \mathbb{N}$, $2 \leq m \leq n$, a commutative subalgebra $\mathcal{B} \subseteq M_m(\mathbb{F})$ of the length $m-2$ and of the form $\mathbb{F}E + \mathcal{N}$, where $\mathcal{N}$ is a nilpotent algebra, and if $m < n$, a commutative subalgebra $\mathcal{C} \subseteq M_{n-m}(\mathbb{F})$ generated by a non-derogatory matrix, such that the algebra $\mathcal{A}$ is conjugated with the algebra $\mathcal{B} \oplus \mathcal{C}$.*

Applying the description of some nilpotent commutative subalgebras in $M_n(\mathbb{F})$ given by D.A. Suprunenko, R.I. Tyshkevich [5, Chapter 3] and I.A. Pavlov [6], we obtain our main result:

Theorem 2. *Let $n \geq 3$ and let $\mathbb{F}$ be a field with at least $n+1$ elements. Consider the matrix $A = E_{1,2} + \cdots + E_{n-2,n-1}$, where $E_{i,j}$ denotes the (i,j)-th matrix unit. Let $\mathcal{A}$ be a commutative subalgebra in $M_n(\mathbb{F})$, which contains the identity matrix I_n. Then $l(\mathcal{A}) = n-2$ if and only if the algebra $\mathcal{A}$ is conjugated in $M_n(\mathbb{F})$ with one of the following algebras:*

1. $\mathbb{F}I_2 \oplus \mathcal{C}_{n-2}$, where $\mathcal{C}_{n-2} \subset M_{n-2}(\mathbb{F})$ is a subalgebra generated by a non-derogatory matrix;

2. $\mathcal{A}_{0;n} = \langle I_n, A, A^2, \ldots, A^{n-2}\rangle$;

3. $\mathcal{A}_{1;n} = \langle E_{1,n},\ C |\ C \in \mathcal{A}_{0;n}\rangle$;

4. $\mathcal{A}_{2;n} = \langle E_{n,n-1},\ C |\ C \in \mathcal{A}_{0;n}\rangle$;

5. if $n = 4$, $\mathcal{A}_{3;4}(\alpha) = \langle E_{1,4} + \alpha E_{4,3},\ C |\ C \in \mathcal{A}_{0;4}\rangle$, $\alpha \in \mathbb{F} \setminus \{0\}$;

6. if $n = 4$, $\mathrm{char}\mathbb{F} = 2$, $\mathcal{A}_{4;4} = \langle E_4, E_{1,2} + E_{3,4},\ E_{1,3} + E_{2,4},\ E_{1,4}\rangle$;

7. if $n = 4$, $\mathrm{char}\mathbb{F} = 2$, $\mathcal{A}_{5;4}(\beta) = \left\{ \begin{pmatrix} aC(\beta) + bI_2 & cC(\beta) + dI_2 \\ O & aC(\beta) + bI_2 \end{pmatrix} | a, b, c, d \in \mathbb{F} \right\}$, where $C(\beta)$ is a companion matrix of an irreducible polynomial $t^2 + \beta \in \mathbb{F}[t]$;

8. $\mathcal{A}_{j;m} \oplus \mathcal{C}_{n-m}$, where $j = 0, 1, 2$, $3 \leq m < n$, $\mathcal{C}_{n-m} \in M_{n-m}(\mathbb{F})$ is a subalgebra generated by a non-derogatory matrix.

Algebras of types 2–7 are pairwise non-conjugate.

This result is a generalization of the similar classification obtained for algebras over algebraically closed fields in [7].

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